CLASS: Msc MATHEMATICS II SEM

SUBJECT CODE: H-2051

SUBJECT NAME: ADVANCED DISCRETE MATHEMATICS

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(01) CH-03 Lattices | ADM H-2051 | Dr Satish Kr Gupta

Let $(L, \leq)$ be a poset. Then $(L, \leq)$ is called a lattice if $\forall\ a, b \in L$

(1) $\sup(a, b) \in L$

and (2) $\inf(a, b) \in L$.

Note. (1) $a \vee b = \sup(a, b)$ OR $(a+b) = \sup$ of a & b

(2) $a \wedge b = \inf(a, b)$ OR $a \cdot b = \inf$ of a & b

(1) other notations of sup a & b $= a \cup b$

inf a & $b = a \cap b$

(2) if $(L, |)$ is a poset then $\forall\ a, b \in L$

sup a & b = lcm of a and b

inf a and b = gcd of a & b etc.

Example of lattice

(1) Let $X = \{1, 2, 3\}$ then

$\{P(X) = \{\phi, (1), (2), (3); (12), (13), (23), (123)\}, \subseteq\}$

is a lattice as $\forall\ a, b \in P(X)$

sup of a and b = $a \cup b \in P(X)$

Inf of a & b = $a \cap b \in P(X)$

(2) Let $\{A = \{2, 3, 4, 6\}, |\}$ be a poset

but $(A, |)$ is not a lattice.

as sup 3 and 4 = lcm of 3 and 4

$= 12$ but $12 \notin A$

sup. 4 & 6 $= 12$ but $12 \notin A$

$\Rightarrow (A, |)$ is not a lattice

(3) Let $\{L = \{1, 2, 3, 4, 6, 12\}, |\}$ be a poset

it is also a lattice as $\forall\ a, b \in L$

sup a & b $\in L$

and inf a & b $\in L$

Take $a = 3$, $b = 12$ sup 3 & 12 = l.c.m of 3 & 12 $= 12 \in L$

inf 3 & 12 = 3 & $3 \in L$

Let $a = 2$, $b = 3$

sup 2 & 3 = l.c.m of 2 & 3 = 6 & $6 \in L$

inf 2 & 3 = g.c.d of. 2 & 3 = 1 & $1 \in L$

We observe take any pair of elements of L, we get sup of $(a, b) \in L$ and inf of $(a, b) \in L$.

$\Rightarrow (L, |)$ is a lattice.

(4) Let $(N, \leq)$ be a poset

it is also lattice as $\forall\ a, b \in N$

sup a and b = a if $a \geq b$ [$\because$ sup$(a, b) = \max(a, b)$]

= b if $b \geq a$

Inf a & b = a if $a \leq b$ [$\because$ Inf$(a, b) = \min(a, b)$]

= b if $b \leq a$.

Some properties of Lattices

Let $(L, \leq)$ be a lattice then the following hold:

(1) (a) $a \wedge a = a$ (b) $a \vee a = a$ (Idempotent law)

(2) (a) $a \wedge b = b \wedge a$ (b) $a \vee b = b \vee a$ (Commutative law)

(3) (a) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ (b) $(a \vee b) \vee c = a \vee (b \vee c)$ (Associativity)

(4) (a) $a \wedge (a \vee b) = a$ (b) $a \vee (a \wedge b) = a$ Absorption law

Theorem. 1. Let $(L, \leq)$ be a lattice in which $\vee$ and $\wedge$ denotes the operations namely join and meet respectivily. Then For any $a, b \in L$ prove

(a) $a \leq b \iff a \wedge b = a$

(b) $a \leq b \iff a \vee b = b$

Proof. Let $(L, \leq)$ be a lattice having $\vee$ and $\wedge$ as join and meet respectively

(a) <u>If part</u> suppose $a \wedge b = a$

to prove $a \leq b$

It is given

$$a \wedge b = a.$$

We know

$$a \wedge b = \inf a \,\&\, b$$
$$= \text{g.l.b of } a \,\&\, b$$

$$\Rightarrow a \wedge b \leq b$$
$$\Rightarrow a \leq b \quad [\because a \wedge b = a]$$

<u>"Only if part"</u> suppose $a \leq b$ —— (1)

To prove $a \wedge b = a$.

It is given $a \leq b$ —— (2)

Also we know

$a \leq a$ —— (3) $[\because \leq \text{ is reflexive}]$

(2) and (3) $\Rightarrow$

a is l.b of a and b

$\Rightarrow a \leq$ g.l.b of a and b

$\Rightarrow a \leq \inf(a, b)$ —— (4)

$\Rightarrow a \leq a \wedge b$ —— (5)

but $a \wedge b \leq a$

$\because$ (4) and (5) $\Rightarrow$

$$a \wedge b = a$$

(b) let $(L, \leq)$ be a lattice

"If part" Suppose $a \vee b = b$

To prove $a \leq b$

We know

$$a \vee b = \sup a \text{ and } b$$

$\Rightarrow a \vee b \geq a$ $\quad [\because a \vee b \geq a$ or $a \vee b \geq b]$

$\Rightarrow b \geq a$ $\quad [\because a \vee b = b]$

$\Rightarrow a \leq b$

"only if part" Suppose $a \leq b$

To prove $a \vee b = b$.

It is given

$b \geq a$ ——— (1)

Also we know

$b \geq b$ ——— (2)

(1) & (2) $\Rightarrow$

b is u.b of a and b

$\Rightarrow b \geq a \vee b$ ——— (3)

but

$a \vee b \geq b$

i.e $b \leq a \vee b$ ——— (4)

from (3) & (4), we have

$$a \vee b = b$$

Remember (1) if $a \leq b$ then $a \wedge b = a$
$a \vee b = b$

(2) $a \wedge b \leq a$ and $a \wedge b \leq b$

(3) $a \vee b \geq a$ and $a \vee b \geq b$

Theorem. 02. Let $(L, \leq)$ be a lattice. Let $a, b, c \in L$ then

(a) $a \leq b$ and $a \leq c \Rightarrow a \leq b \vee c$

(b) $a \leq b$ and $a \leq c \Rightarrow a \leq b \wedge c$

Proof. Let $(L, \leq)$ be a lattice. Let $a, b, c \in L$

(a) Let $a \leq b$ ——— (1)
$a \leq c$ ——— (2)

We know that
$b \vee c \geq b$ ——— (3)
and $b \geq a$, from (1) ——— (4)

$\Rightarrow b \vee c \geq b \geq a$ [from (3) and (4)]
$\Rightarrow b \vee c \geq a$
$\Rightarrow a \leq b \vee c$, proved

(b) It is given $a \leq b$ ——— (5)
and $a \leq c$ ——— (6)

$\Rightarrow a$ is l.b of b and c
$\Rightarrow a \leq$ g.l.b of b and c
$\Rightarrow a \leq (b \wedge c)$, proved

Theorem. 03. Let $(L, \leq)$ be a lattice. Let $a, b, c \in L$

To prove

(a) $a \leq b$ and $c \leq d \Rightarrow a \vee c \leq b \vee d$

(b) $a \leq b$ and $c \leq d \Rightarrow a \wedge c \leq b \wedge d$

Proof. (a) Let $(L, \leq)$ be a lattice. Let $a, b, c \in L$.

It is given

$a \leq b$ ——— (1)

$c \leq d$ ——— (2)

To prove $a \vee c \leq b \vee d$

We know

$b \vee d \geq b$ ——— (3)

and $b \vee d \geq d$ ——— (4)

(1) and (3) $\Rightarrow$

$b \vee d \geq b \geq a \Rightarrow b \vee d \geq a$ — (5)

(2) and (4) $\Rightarrow$

$b \vee d \geq d \geq c \Rightarrow b \vee d \geq c$ — (6)

(5) and (6) $\Rightarrow$

$b \vee d$ is ub of a and c

$\Rightarrow b \vee d \geq$ lub of a and c

$\Rightarrow b \vee d \geq a \vee c$

$\Rightarrow (a \vee c) \leq (b \vee d)$ ——— (7).

(b) Let $(L, \leq)$ be a lattice. Let $a, b, c \in L$

It is given

$a \leq b$ ——— (8)

$c \leq d$ ——— (9)

To prove $a \wedge c \leq b \wedge d$

We know that

$a \wedge c \leq a$ ——— (10)

$a \wedge c \leq c$ ——— (11)

(8) & (10) $\Rightarrow$ $a \wedge c \leq a \leq b \Rightarrow a \wedge c \leq b$ ——— (12)

(9) & (11) $a \wedge c \leq c \leq d \Rightarrow a \wedge c \leq d$ ——— (13)

(12) and (13) $\Rightarrow$

$(a\wedge c)$ is l.b of b and d

$\Rightarrow (a\wedge c) \le$ g.l.b of b and d

$\Rightarrow a\wedge c \le b\wedge d$, proved.

Theorem. 04. Let $(L, \le)$ be a lattice. Let $a, b, c \in L$ Then prove that

(a) $a\wedge(b\vee c) \geq (a\wedge b) \vee (a\wedge c)$

(b) $a\vee(b\wedge c) \le (a\vee b) \wedge (a\vee c)$

Proof. Let $(L, \le)$ be a lattice. Let $a, b, c \in L$

To prove

(a) $a\wedge(b\vee c) \geq (a\wedge b)\vee(a\wedge c)$

We know that

$a\wedge b \le a$ ——— (1)

Also $a\wedge b \le b$ and $b \le b\vee c$ $[\because b\vee c \geq b]$

$\Rightarrow a\wedge b \le b\vee c$ ——— (2)

(1) and (2) $\Rightarrow$

$a\wedge b$ is l.b of a and $(b\vee c)$

$\Rightarrow a\wedge b \le$ g.l.b of a and $(b\vee c)$

$a\wedge b \le a\wedge(b\vee c)$ ——— (3)

Again, $a\wedge c \le a$ ——— (4)

$a\wedge c \le c$ and $c \le b\vee c$

$\Rightarrow a\wedge c \le b\vee c$ ——— (5)

(4) & (5) $\Rightarrow$

$a\wedge c \le a\wedge(b\vee c)$ ——— (6)

(5) and (6) $\Rightarrow$

$a \wedge (b \vee c)$ is ub of $(a \wedge b)$ and $(a \wedge c)$

$\Rightarrow a \wedge (b \vee c) \geq$ lub of $(a \wedge b)$ and $(a \wedge c)$

$\Rightarrow a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

(b) To prove

$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

We know that

$a \vee b \geq a$ ——— (1)

$a \vee b \geq b$ and $b \geq (b \wedge c)$

$\Rightarrow a \vee b \geq (b \wedge c)$ ——— (2)

$\Rightarrow a \vee b$ is ub of a and $(b \wedge c)$

$\Rightarrow a \vee b \geq$ lub of a & $(b \wedge c)$

$\Rightarrow a \vee b \geq a \vee (b \wedge c)$ ——— (3)

Again

$a \vee c \geq a$

$a \vee c \geq c \geq (b \wedge c)$

$\Rightarrow a \vee c$ is ub of a and $(b \wedge c)$

$\Rightarrow a \vee c \geq$ lub of a & $(b \wedge c)$

$\Rightarrow a \vee c \geq a \vee (b \wedge c)$ ——— (4)

(3) & (4) $\Rightarrow$

$a \vee (b \wedge c)$ is l.b of $(a \vee b)$ and $(a \vee c)$

$\Rightarrow a \vee (b \wedge c) \leq$ g.l.b of $(a \vee b)$ and $(a \vee c)$

$\Rightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$, proved

Modular inequality: Let $(L, \le)$ be a lattice.

Let $a, b, c \in L$

if $a \le c \iff a \vee (b \wedge c) = (a \vee b) \wedge c$

This inequality is called modular inequality. [$\because a \vee c = c$ if $a \le c$]

Question 1. Show in a lattice if absorption law holds then Idempotent law will exist

Ans Let in a lattice

$$a \wedge (a \vee b) = a \quad \text{——— (1)}$$

put $a \wedge b$ for b in (1)

$$a \wedge [a \vee \{a \wedge b\}] = a.$$

$$\Rightarrow a \wedge [a \vee (a \wedge b)] = a$$

$$\Rightarrow a \wedge a = a \quad [\because a \vee (a \wedge b) = a]$$

Again

$$a \vee (a \wedge b) = a \quad \text{——— (2)}$$

replace b by $(a \vee b)$ in (2)

$$a \vee [a \wedge (a \vee b)] = a$$

$$\Rightarrow a \vee [a] = a \quad [\because a \wedge (a \vee b) = a]$$

$$\Rightarrow a \vee a = a$$

This shows Idempotent law follow if absorption law is there.

Theorem. 06. Let $(L, \leq)$ be a lattice. Let $a, b, c, d \in L$ then prove that

(a) $(a \wedge b) \vee (c \wedge d) \leq (a \vee c) \wedge (b \vee d)$

(b) $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \leq (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$

Proof. Let $(L, \leq)$ be a lattice. Let $a, b, c, d \in L$

(a) part To prove

$$(a \wedge b) \vee (c \wedge d) \leq (a \vee c) \wedge (b \vee d)$$

We know that

$$a \wedge b \leq a$$
$$c \wedge d \leq c$$

$$\Rightarrow (a \wedge b) \vee (c \wedge d) \leq (a \vee c) \quad \text{——— (1)} \quad \left[\because \begin{array}{l} a \leq b \\ c \leq d \\ \Rightarrow a \vee c \leq b \vee d \end{array} \right]$$

Again,

$$a \wedge b \leq b$$
$$c \wedge d \leq d$$

$$\Rightarrow (a \wedge b) \vee (c \wedge d) \leq (b \vee d) \quad \text{——— (2)}$$

(1) & (2) $\Rightarrow$
$(a \wedge b) \vee (c \wedge d)$ is l.b. of $(a \vee c)$ and $(b \vee d)$

$\Rightarrow (a \wedge b) \vee (c \wedge d) \leq$ g.l.b of $(a \vee c)$ and $(b \vee d)$

$$\boxed{(a \wedge b) \vee (c \wedge d) \leq (a \vee c) \wedge (b \vee d)}$$

Proof. (b) part Let $(L, \leq)$ be a lattice. Let $a, b, c, d \in L$.

To prove $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \leq (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$

We know that

$$a \wedge b \leq a \vee b \quad \text{——— (3)}$$
$$a \wedge b \leq b \leq b \vee c$$
$$\Rightarrow a \wedge b \leq b \vee c \quad \text{——— (4)}$$

also

$$a \wedge b \leq a \leq (c \vee a)$$
$$\Rightarrow a \wedge b \leq (c \vee a) \quad \text{——— (5)}$$

(3), (4) and (5) $\Rightarrow$

$(a\wedge b)$ is a l.b of $(a\vee b)$, $(b\vee c)$ and $(c\vee a)$

$\Rightarrow (a\wedge b) \le$ g.l.b of $(a\vee b)$, $(b\vee c)$ and $(c\vee a)$

$\Rightarrow (a\wedge b) \le (a\vee b)\wedge(b\vee c)\wedge(c\vee a)$ ——— (6)

Similarly

$(b\wedge c) \le (a\vee b)\wedge(b\vee c)\wedge(c\vee a)$ ——— (7)

and $(c\wedge a) \le (a\vee b)\wedge(b\vee c)\wedge(c\vee a)$ ——— (8)

(6), (7) & (8) $\Rightarrow$

$(a\vee b)\wedge(b\vee c)\wedge(c\vee a)$ is ub of $(a\wedge b)$, $(b\wedge c)$ and $(c\wedge a)$

$(a\vee b)\wedge(b\vee c)\wedge(c\vee a) \geq$ lub of $(a\wedge b)$, $(b\wedge c)$ and $(c\wedge a)$

$\Rightarrow (a\vee b)\wedge(b\vee c)\wedge(c\vee a) \geq (a\wedge b)\vee(b\wedge c)\vee(c\wedge a)$

Theorem. 07. Prove that dual of a lattice is lattice

Solution. Let (L, R) be a lattice

Let $(L, \bar{R})$ be dual of (L, R)

To prove $(L, \bar{R})$ is also a lattice

To prove $(L, \bar{R})$ and $\inf(x,y) \in L$

Let $x, y \in L \Rightarrow \sup(x,y) \in L$

We know

$x\vee y = \sup(x,y)$ in (L, R) ——— (1)

$\Rightarrow x\,R\,(x\vee y)$ ——— (2)

and $y\,R\,(x\vee y)$

(1) $\Rightarrow x\,R\,(x\vee y) \Rightarrow (x\vee y)\,\bar{R}\,x$ ——— (3)

(2) $\Rightarrow y\,R\,(x\vee y) \Rightarrow (x\vee y)\,\bar{R}\,y$ ——— (4)

(3) and (4) $\Rightarrow$

$(x\vee y)$ is a l.b of x and y in $(L, \bar{R})$

To show

$(x \vee y)$ is g l b of x and y in $(L, \bar{R})$

Let z be any l. b of x and y in $(L, \bar{R})$

$\Rightarrow z \bar{R} x$ ——— (5)

and $z \bar{R} y$ ——— (6)

(5) $\Rightarrow z \bar{R} x \Rightarrow x R z$ — (7)

(6) $\Rightarrow z \bar{R} y \Rightarrow y R z$ — (8)

(7) and (8) $\Rightarrow$

z is u.b. of. x & y in (L, R)

$\Rightarrow (x \vee y) R z$ [$\because x \vee y$ = sup of x & y = l u b of x & y]

$\Rightarrow z \bar{R} (x \vee y)$

$\Rightarrow (x \vee y)$ is g.l b of x & y in $(L, \bar{R})$

$\Rightarrow (x \vee y) = \inf(x, y)$ in $(L, \bar{R})$

Similarly

$x \wedge y = \sup(x, y)$ in $(L, \bar{R})$

$\therefore (L, \bar{R})$ is a lattice.

Proved

Lattice as algebraic systems (Second def. of lattice)

A non-empty set L with two binary operations $\wedge$ and $\vee$ is said to be lattice if following properties are satisfied:

1. Commutativity. Let $a, b \in L$
   $a \wedge b = b$ and $a \vee b = b \vee a$

2. Associativity. Let $a, b, c \in L$ then
   (a) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$
   and (b) $a \vee (b \vee c) = (a \vee b) \vee c$

3. Absortion property. Let $a, b \in L$ then
   (a) $a \vee (a \wedge b) = a$
   and (b) $a \wedge (a \vee b) = a$

Note. $(L, \wedge, \vee)$ is a lattice if it follows CAA
C $\Rightarrow$ Commutative, A = Associative, A = Absorption

Sublattices. Let $(L, \wedge, \vee)$ be a lattice. Let $S \subseteq L$ Then S is called sublattice if $(S, \wedge, \vee)$ is itself a lattice under the same operations as that in L.

ex Let $\{L = P(X) = \{\phi, (a), (b), (c), (a, b), (a, c), (b, c), (a, b, c)\}, \cup, \cap\}$ be a lattice $X = (a, b, c)$

then $S = \{\phi, (a), (b), (a\ b)\}$ is a sublattice of $P(X)$.

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Direct Product of lattice

Theorem. Let $(L_1, \wedge, \vee)$ and $(L_2, \wedge, \vee)$ be any lattices then show $(L_1 \times L_2, \wedge, \vee)$ is also a lattice where $L_1 \times L_2 = \{(a, b) \mid a \in L_1, b \in L_2\}$, $\wedge$ and $\vee$ are defined as

$$(a_1, b_1) \wedge (a_2, b_2) = (a_1 \wedge a_2, b_1 \wedge b_2)$$

and $(a_1, b_1) \vee (a_2, b_2) = (a_1 \vee a_2, b_1 \vee b_2)$.

Proof. To show $\{L_1 \times L_2, \wedge, \vee\}$ is a lattice

1. commutativity. Let $(a_1, b_1), (a_2, b_2) \in L_1 \times L_2$

(a) $(a_1, b_1) \wedge (a_2, b_2) = (a_1 \wedge a_2, b_1 \wedge b_2)$

$$= (a_2 \wedge a_1, b_2 \wedge b_1) \quad \left[\because \begin{array}{l} a_1 \wedge a_2 = a_2 \wedge a_1 \\ b_1 \wedge b_2 = b_2 \wedge b_1 \end{array}\right]$$

$$= (a_2, b_2) \wedge (a_1, b_1)$$

(b) $(a_1, b_1) \vee (a_2, b_2) = (a_1 \vee a_2, b_1 \vee b_2)$

$$= (a_2 \vee a_1, b_2 \vee b_1) \quad \left[\because \begin{array}{l} a_1 \vee a_2 = a_2 \vee a_1 \\ b_1 \vee b_2 = b_2 \vee b_1 \end{array}\right]$$

$$= (a_2, b_2) \vee (a_1, b_1)$$

2. Associativety. Let $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in L_1 \times L_2$

To show

(a) $(a_1, b_1) \wedge [(a_2, b_2) \wedge (a_3, b_3)] = [(a_1, b_1) \wedge (a_2, b_2)] \wedge (a_3, b_3)$

$$LHS = (a_1, b_1) \wedge [(a_2, b_2) \wedge (a_3, b_3)]$$

$$= (a_1, b_1) \wedge (a_2 \wedge a_3, b_2 \wedge b_3)$$

$$= [a_1 \wedge (a_2 \wedge a_3), b_1 \wedge (b_2 \wedge b_3)]$$

$$= [(a_1 \wedge a_2) \wedge a_3, (b_1 \wedge b_2) \wedge b_3]$$

$$= [(a_1, b_1) \wedge (a_2, b_2)] \wedge (a_3, b_3) = RHS$$

Similarly we can show:

$(a_1, b_1) \vee [(a_2, b_2) \vee (a_3, b_3)] = [(a_1, b_1) \vee (a_2, b_2)] \vee [a_3, b_3]$

Absorption. Let $(a_1, b_1), (a_2, b_2) \in L_1 \times L_2$

To show $(a_1, b_1) \vee [(a_1, b_1) \wedge (a_2, b_2)] = (a_1, b_1)$

$LHS = (a_1, b_1) \vee [(a_1, b_1) \wedge (a_2, b_2)]$

$= (a_1, b_1) \vee [a_1 \wedge a_2, b_1 \wedge b_2]$

$= [a_1 \vee (a_1 \wedge a_2), b_1 \vee (b_1 \wedge b_2)]$

$= (a_1, b_1)$

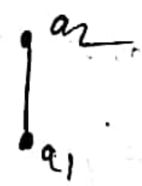
$= RHS$

Similarly

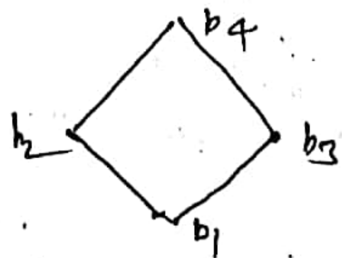
$(a_1, b_1) \wedge [(a_1, b_1) \vee (a_2, b_2)] = (a_1, b_1)$

Therefore, $\{L_1 \times L_2, \wedge, \vee\}$ is a lattice.

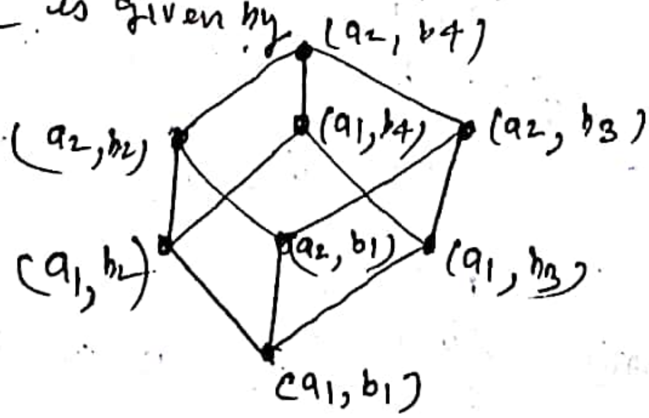
Example: Let L_1 is



L_2 is



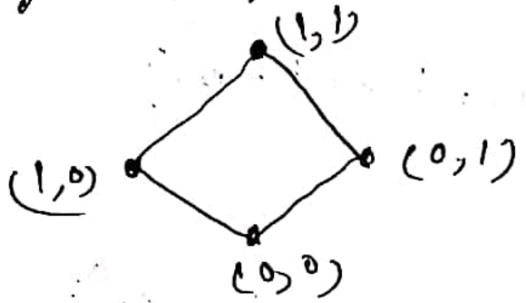
than $L_1 \times L_2$ is given by



Example: Let $L = (0,1)$ be a chain then draw L^2 as Hasse-diagram.

$$L^2 = L \times L = (0,1) \times (0,1)$$
$$= \{(0,0), (0,1), (1,0), (1,1)\}$$

Hasse diagram of $L \times L$ is as under



Isomorphic lattices

Let $(L_1, \wedge, \vee)$ and $(L_2, \wedge, \vee)$ be any two lattices then a mapping

$f : L_1 \longrightarrow L_2$ is called isomorphism

if (1) f is one-one (2) f is onto

(3) $f(a \wedge b) = f(a) \wedge f(b)$
$f(a \vee b) = f(a) \vee f(b)$, $\forall a, b \in L_1$.

Example. $\{L_1 = \{1, 2, 3, 6\}, |\}$ is a lattice

$L_2 = P(S) = \{\phi, \{a\}, \{b\}, \{a,b\}\}$ be another lattice. Then

$f : L_1 \longrightarrow L_2$ is an isomorphism if

$f(1) = \phi$, $f(2) = \{a\}$, $f(3) = \{b\}$

$f(6) = \{a,b\}$.

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$\because$ We can varify (a) $f(a \wedge b) = f(a) \wedge f(b)$
(b) $f(a \vee b) = f(a) \vee f(b)$
$\forall a, b \in L_1$

Take $a = 1$, $b = 2$,

$f(1 \wedge 2) = f(1)$ [$\because 1 \wedge 2 = $ g.c.d of 1 & 2 $= 1$]

$= \phi$ —(1) [$\because f(1) = \phi$

Now

$f(1) \wedge f(2) = \phi \wedge (a)$

$= \phi$ —(2)

$\because$ (1) & (2) $\Rightarrow$

$$\boxed{f(1 \wedge 2) = f(1) \wedge f(2)}$$

$f(1 \vee 2) = f($ l.c.m of 1 and 2$)$

$= f(2)$

$= (a)$ —— (3)

$f(1) \vee f(2) = \phi \vee (a)$

$= (a)$ —— (4)

$\because$ (3) & (4) $\Rightarrow$

$$\boxed{f(1 \vee 2) = f(1) \vee f(2)}$$

Similarly we can go for other values of L_1.

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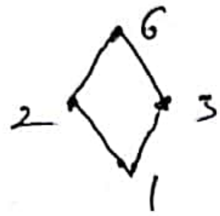
Bounded lattice.

A lattice $(L, \leq)$ is called bounded if L contains its greatest element and least element.

Example. $\{L = \{1, 2, 3, 6\}, |\}$ is a poset



Here L is bounded lattice as

L has its least element as 1

and L has its greatest element is 6.

Note: Least element is called universal lower bound.
greatest element is called universal upper bound.

Theorem. Prove that every finite lattice is a bounded lattice.

Proof. Let $\{L = \{a_1, a_2, \cdots, a_n\}, \wedge, \vee\}$ be a finite lattice

To show L contains least element and greatest element.

Put
$$b_1 = a_1$$
$$b_2 = a_2 \wedge b_1$$
$$b_3 = a_3 \wedge b_2$$
$$\vdots$$
$$b_n = a_n \wedge b_{n-1}$$

$\Rightarrow b_1, b_2, \ldots b_n$ are belong to L

$\Rightarrow b_n \leq a_i, \ \forall \ i = 1, 2, 3 \ldots, n$

$\Rightarrow b_n$ is the least element of L

& $b_n = a_1 \wedge a_2 \wedge a_3 \cdots \wedge a_n$

Similarly $a_1 \vee a_2 \vee \cdots \vee a_n$ is the greatest element of L.

$\Rightarrow (L, \wedge, \vee)$ is bounded lattice.

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Theorem. 02. Let $(L, \leq)$ be a lattice with least element 0 and the greatest element 1. For any $a \in L$ show

(a) $a \vee 1 = 1$ and $a \wedge 1 = a$

(b) $a \vee 0 = a$ and $a \wedge 0 = 0$

Proof. (a) Let $a \in L$, 1 is the greatest element of L

$\Rightarrow \quad a \vee 1 \leq 1$ ——— (1)

since $a \vee 1 =$ sup of a and 1

$\Rightarrow a \vee 1 \geq 1$ ——— (2)

from (1) and (2), we have

$\boxed{a \vee 1 = 1}$

Again $a \wedge 1$ is the infimum of a & 1

$\Rightarrow \quad a \wedge 1 \leq a$ ——— (3)

Also let $a \leq a$

$a \leq 1$

$\Rightarrow a$ is lb of a & 1

$\Rightarrow a \leq$ glb of a & 1

$\Rightarrow a \leq a \wedge 1$ ——— (4)

(3) & (4) $\Rightarrow$

$\boxed{a \wedge 1 = a}$

(b) Let $a \in L$, 0 is the least element of L

$\because \; 0 \leq a$

and $a \leq a$

$\Rightarrow a$ is ub of 0 and a

$\Rightarrow a \geq$ lub of 0 & a

$\Rightarrow a \geq 0 \vee a$ ——— (5)

but

$0 \vee a \preceq a$ ——— (6)

[∵ $0 \vee a$ = sup of 0 and a]

(5) & (6) ⟹

$\boxed{0 \vee a = a}$

Again

$0 \preceq a \wedge 0$ ——— (7)

[∵ 0 is least element of L]

we know

$a \wedge 0$ = inf of a & 0

$\Rightarrow a \wedge 0 \preceq 0$ ——— (8)

(7) & (8) ⟹

$\boxed{a \wedge 0 = 0}$

Complete lattice. A lattice $(L, \wedge, \vee)$ is called complete if every non-empty subset of L has its supremum and infimum in L.

Theorem. Prove that dual of a complete lattice is complete.

Proof. Let (L, R) be a complete lattice

let $(L, \bar{R})$ be the dual of (L, R)

To prove $(L, \bar{R})$ is also complete lattice

let $S \subseteq L$

it is given (L, R) is complete

$\Rightarrow$ S has its sup and inf in (L, R)

let $a \in \inf S$

$\Rightarrow a R x, \ \forall x \in L$

$\Rightarrow x \bar{R} a$, $\forall x \in L$

$\Rightarrow a$ is ub of S in $(L, \bar{R})$

Let b be any other U.b of S in $(L, \bar{R})$

$\Rightarrow x \bar{R} b$, $\forall x \in S$ in $(L, \bar{R})$

$\Rightarrow b R x$, $\forall x \in S$ in (L, R)

put $x = a$

$\Rightarrow b R a$

$\Rightarrow a \bar{R} b$

$\Rightarrow a$ is lub of S in $(L, \bar{R})$

$\Rightarrow a =$ sup of S in $(L, \bar{R})$

Similarly we can prove

if $a \in$ sup of S in (L, R)

$\Rightarrow a \in$ inf of S in $(L, \bar{R})$

$\Rightarrow$ Every ~~subset~~ S of L has its sup and inf in $(L, \bar{R})$

$\Rightarrow (L, \bar{R})$ is also a lattice.

Hence dual of a complete lattice is complete lattice.

Complement of an element of a lattice

Let $(L, \wedge, \vee)$ be a bounded lattice with least element 0 and greatest element 1. Let $a, b \in L$ then b is called complement of a if $a \vee b = 1$ and $a \wedge b = 0$

i.e $a \vee a' = 1$ and $a \wedge a' = 0$ OR $a' = b$ i.e $a \vee b = 1$, $a \wedge b = 0$

Here $b = a' =$ complement of a.

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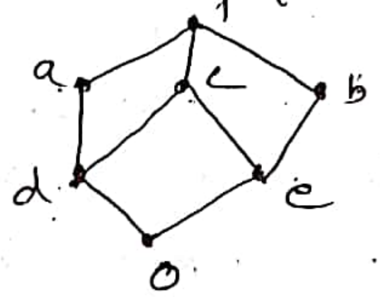
Note. complement need not be unique



Here 1 is greatest element
0 is least element

Here complements of a are b and e

as $a \vee b = 1$ $\quad a \wedge b = 0$

$a \vee e = 1$ $\quad a \wedge e = 0$

[d & e; दोनों के l.b नहीं है?]

Note: c has no complement as

$c \vee a = 1$ but $c \wedge a = d$ which is not least

$c \vee b = 1$ but $c \wedge b = e$

<u>Complemented lattice</u> A bounded lattice L is said to be complemented lattice if every element in L has a complement.

Theorem. Prove that dual of a complemented lattice is complemented lattice.

Proof. Let (L, R) be a complemented lattice

Let $(L, \bar{R})$ be the dual of (L, R).

To prove $(L, \bar{R})$ is also complemented lattice

Let $a \in L$. Let a' be the complement of a

then $\underset{(i)}{a \vee a' = 1}$ and $\underset{(ii)}{a \wedge a' = 0}$

(i) $\Rightarrow$ $1 = \sup a$ and a' in (L, R)

$\Rightarrow a R 1 \Rightarrow 1 \bar{R} a$

and $a' R 1 \Rightarrow 1 \bar{R} a'$

$\Rightarrow 1$

$\Rightarrow 1$ is l.b of a & a' in $(L, \bar{R})$

To prove 1 is g.l.b of a & a' in $(L, \bar{R})$

let b be another l.b of a & a' in $(L, \bar{R})$

$\Rightarrow b\bar{R}a \Rightarrow aRb$

and $b\bar{R}a' \Rightarrow a'Rb$

$\Rightarrow b$ is u.b of a and a' in (L, R)

$\Rightarrow a \vee a' R b$ [$\because a \vee a' =$ lub of a & a']

$\Rightarrow 1 R b$ [$\because a \vee a' = 1$]

$\Rightarrow b \bar{R} 1$

$\Rightarrow 1$ is g.l.b in $(L, \bar{R})$

$\Rightarrow 1 =$ inf of a & a' in $(L, \bar{R})$

Again, let $a \wedge a' = 0$ in (L, R)

$\Rightarrow 0 R a$ [$\because 0$ is inf of a & a']

and $0 R a'$

$0 R a \Rightarrow a \bar{R} 0$

$0 R a' \Rightarrow a' \bar{R} 0$ in $(L, \bar{R})$

$\Rightarrow 0$ is ub of a and a' in $(L, \bar{R})$

To prove 0 is lub of a & a' in $(L, \bar{R})$

let b be another ub of a & a' in $(L, \bar{R})$

$\Rightarrow a\bar{R}b$

& $a'\bar{R}b$

$\Rightarrow a \wedge a' \bar{R} b$

$\Rightarrow 0 \bar{R} b$

$\Rightarrow 0$ is lub of a & a' in $(L, \bar{R})$

Hence $(L, \bar{R})$ is also complemented lattice

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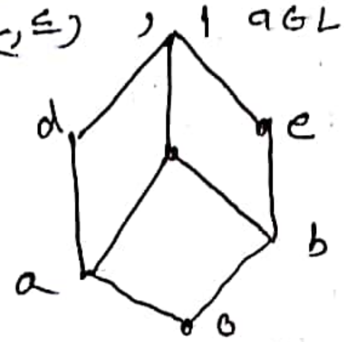
Cover of an element in a lattice

Let a and b be any two elements of a lattice then a is said to cover b if $a > b$ and there is no c such that $a > c > b$.

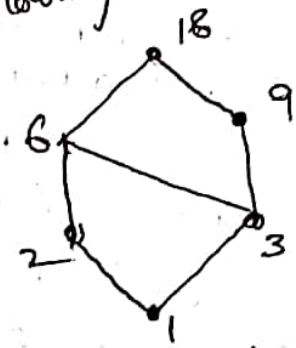
Atom. In a lattice $(L, \leq)$, $a \in L$ is called atom if a covers 0 element [0 element is least element] & $\exists$ no c s.t. $a > c > 0$.

Ex ① Find atom



only a and b are atoms of the above lattice.

② Find atoms of the following



Here 2 and 3 are atoms as 2 covers 1 and 3 covers 1, 1 is the least element.

Note. Here covers of 3 are 6 and 9.

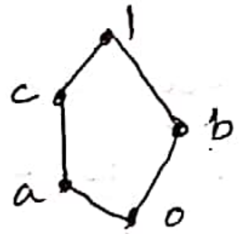
Join irreducible element in a lattice

Let $(L, \leq)$ be a lattice. An element $a \in L$ is said to be join irreducible if $a = x \vee y \Rightarrow a = x$ or $a = y$.

Meet irreducible element in a lattice

Let $(L, \leq)$ be a lattice. An element $a \in L$ is said to be meet irreducible if $a = x \wedge y \Rightarrow a = x$ or $a = y$.

Example



In this lattice a is join irreducible

as $a \vee 0 = a$ or $a \vee c = c$

b is join irreducible as $b \vee 0 = b$ or $b \vee 1 = 1$

c is join irreducible as

$c = a \vee c$

$1 = c \vee 1$

1 is not join irreducible as $1 = x \vee y \Rightarrow$ neither $x = 1$ or $y = 1$

0 is not joint irreducible as $0 = a \wedge b$ neither $a = 0$ nor $b = 0$

Modular lattices

A lattice $(L, \wedge, \vee)$ is said to be lattice if

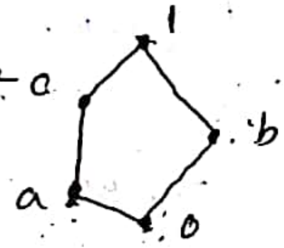
$\forall\ a, b, c \in L$

$a \le c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$.

Example. Every chain is a modular lattice

Note. A totally ordered set is called a chain.
[i.e a poset whose every two elements are comparable]

Example. A pentagonal is not modular lattice



Pentagonal lattice

Here $a \le c$ but

$a \vee (b \wedge c) = a \neq (a \vee b) \wedge c$

Here LHS $= a \vee (b \wedge c) = a \vee 0 = a$ $[b \wedge c = 0]$

RHS $= (a \vee b) \wedge c = 1 \wedge c = c$ $[a \vee b = 1]$

Distributive lattice

A lattice $(L, \wedge, \vee)$ is said to be distributive if for any $a, b, c \in L$

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

OR

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

ex (1) $(P(S), \subseteq)$ is a distributive lattice

(2) let $S = \{1, 2, 3\}$

$P(S) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

then $(P(S), \subseteq)$ is a distributive lattice

because $a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$, $\forall a, b, c \in P(S)$.

Theorem. A lattice $(L, \wedge, \vee)$ is distributive iff

$$(a \vee b) \wedge (b \vee c) \wedge (c \vee a) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$$

Proof. "If part"

suppose $(L, \wedge, \vee)$ be a distributive lattice

To prove $(a \vee b) \wedge (b \vee c) \wedge (c \vee a) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$

$LHS = (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$

$= \{(b \vee c) \wedge (c \vee a)\} \wedge (a \vee b)$ (Commutatively)

$= \{(b \vee c) \wedge (c \vee a)\} \wedge a \vee \{(b \vee c) \wedge (c \vee a)\} \wedge b$ (Distributively)

$= a \wedge [(b \vee c) \wedge (c \vee a)] \vee [b \wedge \{(b \vee c) \wedge (c \vee a)\}]$

$= a \wedge \{(c \vee a) \wedge (b \vee c)\} \vee [b \wedge (b \vee c) \wedge (c \vee a)]$ (Commutatively)

$= \{a \wedge (c \vee a)\} \wedge (b \vee c) \vee [\{b \wedge (b \vee c)\} \wedge (c \vee a)]$ (associatively)

$= a \wedge (b \vee c) \vee [b \wedge (c \vee a)]$

$= [(a \wedge b) \vee (a \wedge c)] \vee [b \wedge c \vee (b \wedge a)]$

$= (a \wedge b) \vee (a \wedge c) \vee (b \wedge c) \vee (b \wedge a)$

$= a \wedge b \vee (b \wedge a) \vee (a \wedge c) \vee (b \wedge c)$

$= (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$ $[\because a \wedge c = c \wedge a$, $b \wedge a = a \wedge b$, $(a \wedge b) \vee (a \wedge b) = a \wedge b]$

$=$ RHS.

Only if part

Suppose

$(a \vee b) \wedge (b \vee c) \wedge (c \vee a) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a)$ —— (1)

To prove $(L, \wedge, \vee)$ is distributive

First we shall prove $(L, \wedge, \vee)$ is modular lattice

let $x, y, z \in L$ such that $x \leq z$ then

$x \vee (y \wedge z) = [x \vee (x \wedge y)] \vee (y \wedge z)$ [x के स्थान पर $x \vee (x \wedge y)$ लिखकर by absorption prop.]

$x \vee (y \wedge z) = (x \wedge z) \vee (x \wedge y) \vee (y \wedge z)$ [Most left (RHS पर) $x = x \wedge z$ as $x \leq z$]

$x \vee (y \wedge z) = (x \vee z) \wedge (x \vee y) \wedge (y \vee z)$, from (1)

$x \vee (y \wedge z) = z \wedge (x \vee y) \wedge (y \vee z)$ [$\because x \vee z = z$ as $x \leq z$]

$x \vee (y \wedge z) = z \wedge (y \vee z) \wedge (x \vee y)$

$x \vee (y \wedge z) = z \wedge (x \vee y)$

$x \vee (y \wedge z) = (x \vee y) \wedge z$

$\Rightarrow (L, \wedge, \vee)$ is modular

We know that

$a \wedge b \leq a$

$a \wedge c \leq a$

$\Rightarrow (a \wedge b) \vee (a \wedge c) \leq a$

let $(a \wedge b) \vee (a \wedge c) = x$

$\Rightarrow \quad x \leq a \quad , x = (a \wedge b) \vee (a \wedge c)$

let $y = (b \wedge c)$

then

$x \vee (y \wedge a) = (x \vee y) \wedge a$, by modular property

i.e.

$\Rightarrow \{(a \wedge b) \vee (a \wedge c)\} \vee [(b \wedge c) \wedge a]$

$= \{[\{(a \wedge b) \vee (a \wedge c)\} \vee (b \wedge c)] \wedge a\}$ — (2)

To prove

$a \wedge (b \vee c) = (a \wedge b) \vee (c \wedge a)$

$= (a \wedge b) \vee (a \wedge c)$

$LHS = a \wedge (b \vee c)$

$= a \wedge (a \vee c) \wedge (b \vee c)$ by Absorption

$= a \wedge (a \vee b) \wedge (a \vee c) \wedge (b \vee c)$ by absorption

$= [(a \vee b) \wedge (a \vee c) \wedge (b \vee c)] \wedge a$ [commutativity]

$a \wedge (b \vee c) = [(a \vee b) \wedge (a \vee c) \wedge (b \vee c)] \wedge a$

$= [(a \wedge b) \vee (a \wedge c) \vee (b \wedge c)] \wedge a$, from (1) (given)

$= \{(a \wedge b) \vee (a \wedge c)\} \vee [b \wedge c \wedge a]$ (from (2))

$= (a \wedge b) \vee \{(c \wedge a) \vee [b \wedge (c \wedge a)]\}$ $(a \wedge c = c \wedge a)$

$a \wedge (b \vee c) = (a \wedge b) \vee (c \wedge a)$, by absorption as $x \vee (b \wedge x) = x$, $x = c \wedge a$

$\Rightarrow (L, \wedge, \vee)$ is a distributive lattice